A dilogarithmic 3-dimensional ising tetrahedron

D.J. Broadhurst

Physics Department, Open University, Milton Keynes MK7 6AA, UK (e-mail: D.Broadhurst@open.ac.uk)

Abstract. In 3 dimensions, the Ising model is in the same universality class as ϕ^4-theory, whose massive 3-loop tetrahedral diagram, C^{Tet}, was of an unknown analytical nature. In contrast, all single-scale 4-dimensional tetrahedra were reduced, in hep-th/9803091, to special values of exponentially convergent polylogarithms. Combining dispersion relations with the integer-relation finder PSLQ, we find that $C^{\mathrm{Tet}}/2^{5/2} = \mathrm{Cl}_2(4\alpha) - \mathrm{Cl}_2(2\alpha)$, with $\mathrm{Cl}_2(\theta) := \sum_{n>0} \sin(n\theta)/n^2$ and $\alpha := \arcsin\frac{1}{3}$. This empirical relation has been checked at 1,000-digit precision and readily yields 50,000 digits of C^{Tet}, after transformation to an exponentially convergent sum, akin to those studied in math.CA/9803067. It appears that this 3-dimensional result entails a polylogarithmic ladder beginning with the classical formula for $\pi/\sqrt{2}$, in the manner that 4-dimensional results build on that for $\pi/\sqrt{3}$.

1 Introduction

In 3 dimensions, the universality class of the Ising model includes ϕ^4 theory, which entails at the 3-loop level a tetrahedral Feynman diagram, corresponding to the symmetrical 9-dimensional integral [1]

$$C^{\mathrm{Tet}} := \frac{1}{\pi^6} \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 d^3\mathbf{k}_3 \Delta(\mathbf{k}_1)\Delta(\mathbf{k}_2)$$
$$\Delta(\mathbf{k}_3)\Delta(\mathbf{k}_1-\mathbf{k}_2)\Delta(\mathbf{k}_2-\mathbf{k}_3)\Delta(\mathbf{k}_3-\mathbf{k}_1) \quad (1)$$

with $\Delta(\mathbf{k}) := 1/(|\mathbf{k}|^2 + 1)$ as the unit-mass propagator. A numerical value, $C^{\mathrm{Tet}} \approx 0.1739006$, was obtained in [2] and checked in [1,3]. We shall show that the dispersive methods of [4,5] enable a reduction of C^{Tet}, as for any assignment of masses, to single integrals of logarithms. Then we shall describe how the lattice algorithm PSLQ [6] achieved a very simple reduction of C^{Tet} to a Clausen integral, which gives an exponentially convergent sum that reveals a new feature of the distinctive mapping [7] of diagrams [5,8–12] to numbers [13–16] provided by quantum field theory.

2 Dispersive integral

Let $C(a,b)$ be the tetrahedron with non-adjacent lines carrying masses a and b, while the other 4 lines retain unit mass. Then a long dispersive calculation produces a short result:

$$C(a,b) = -\frac{16}{b}\int_2^\infty \frac{dw}{(w+a)D(w,b)}\operatorname{arctanh}\left(\frac{N(w,b)}{D(w,b)}\right) \quad (2)$$

where the denominator function

$$D(w,b) := w\sqrt{w^2 + b^2 - 4} \quad (3)$$

is regular at the 2-particle threshold, $w = 2$, provided that $b > 0$, and

$$N(w,b) = w^2 - 2(2+b) \quad \text{for } w \in [2, 2+b] \quad (4)$$
$$N(w,b) = \qquad wb \qquad \text{for } w \in [2+b, \infty] \quad (5)$$

specify a numerator that is continuous in value, though not in derivative, at the 3-particle threshold, $w = 2 + b$. The origins of (2–5) will be outlined, neglecting factors of 2 and π.

1. Let $I(\mathbf{k}, b)$ be the 2-point function obtained by cutting the tetrahedron at the line with mass a, so that

$$C(a,b) \sim \int \frac{d^3\mathbf{k}}{|\mathbf{k}|^2 + a^2} I(\mathbf{k}, b) \quad (6)$$

with the 2-point function given by a dispersion relation of the form

$$I(\mathbf{k}, b) \sim \int_2^\infty \frac{w\,dw}{w^2 + |\mathbf{k}|^2}\sigma(w, b) \quad (7)$$

where σ is the spectral density of I, considered in $2+1$ spacetime dimensions. We perform this anti-Wick rotation, away from the 3 spatial dimensions of condensed matter, in order to exploit the Cutkosky rules of Minkowski-space quantum field theory, as in [4]. An interchange of order of integration in (6,7) gives

$$C(a,b) \sim \int_2^\infty \frac{w\,dw}{w+a}\sigma(w, b) \quad (8)$$

which explains the simple dependence on a of the integrand in (2).

2. The spectral density

$$\sigma(w, b) = \theta(w - 2)\sigma_2(w, b)$$
$$+ \theta(w - 2 - b)\sigma_3(w, b) \qquad (9)$$

receives contributions from intermediate states with 2 and 3 particles. In the first case, $\sigma_2(w, b) \sim \Re F(w + i0, b)/w$ entails a 1-loop form factor, F. This may also be calculated dispersively, from its imaginary part

$$\Im F(w + i0, b) \sim \frac{1}{w} \int_0^\pi \frac{d\phi}{2k^2(1 - \cos\phi) + b^2}$$
$$= \frac{\pi}{wb\sqrt{w^2 + b^2 - 4}} \qquad (10)$$

where $k := \sqrt{(w/2)^2 - 1}$ and ϕ are the centre-of-mass 2-momentum and scattering angle, in the elastic scattering of unit-mass particles, by exchange of a particle of mass b, in $2 + 1$ spacetime dimensions. This is the origin of the square root in (3).

3. It is now straightforward to calculate

$$wb\sigma_2(w, b) \sim \Re \int_2^\infty \frac{x\,dx}{x^2 - w^2 + i0} \frac{1}{D(x, b)} \qquad (11)$$

and obtain logarithms from the real part of the form factor. Maple produced 3 arctanh functions, which were combined, by hand, to give the numerator (4).

4. The 3-particle intermediate state yields the Dalitz-plot integral

$$\sigma_3(w, b) \sim \Re \int_{b(2+b)}^{w(w-2)} \frac{ds}{s} \int \frac{dt}{t} \frac{1}{\sqrt{J(s, t, w^2, b^2)}}$$
$$= \int_{b(2+b)}^{w(w-2)} \frac{ds}{s} \frac{\pi}{\sqrt{-J(s, 0, w^2, b^2)}} \qquad (12)$$

where s and t are the denominators of the propagators of the two particles that are still off-shell and the t integration is over the range in which the Jacobian

$$J(s, t, u, v) := -(s\,t - u\,v)(s + t + 4 - u - v)$$
$$- (s - t)^2 \qquad (13)$$

is positive. Maple produced 2 arctanh functions, to be added to the 3 from σ_2. Manual combination of these 5 logs produced the amazingly simple numerator (5).

This method is clearly generalizable to give a single integral of logs in any mass case.

3 Superconvergence and KLN cancellations

The factor $-16/b$ in (2) looks alarming, at first sight. The integral is manifestly finite as $a \to 0$. Field theory proves that $C(a, b) = C(b, a)$, notwithstanding the very different ways that the masses a and b enter the integral. Hence $C(a, b)$ is finite as $b \to 0$, despite the factor of $1/b$. Already we see that potentially linear infra-red divergences have

been cancelled, by combining 2-particle and 3-particle intermediate states in (5). This parallels the 4-dimensional cancellation of logarithmic divergences, from virtual and real soft photons, by the Kinoshita-Lee-Nauenberg mechanism [17]. However, it is still not safe to take the limit $b \to 0$, blithely, since the contributions from $w > 2 + b$ are manifestly negative, and have a $1/(w - 2)$ singularity as $b \to 0$.

The key to handling this tricky limit is the superconvergence relation

$$0 = \int_2^\infty \frac{dw}{D(w, b)} \operatorname{arctanh}\left(\frac{N(w, b)}{D(w, b)}\right) \qquad (14)$$

which ensures that $\lim_{a \to \infty} aC(a, b) = 0$. Thus one may make the replacement

$$\frac{1}{w + a} \to \frac{1}{w + a} - \frac{1}{2 + a} = -\frac{w - 2}{(w + a)(2 + a)} \qquad (15)$$

in (2). Then the factor $w - 2$ suppresses the singularity at threshold in the limit $b \to 0$, giving the elementary integral

$$C(a, 0) = \frac{16}{2 + a} \int_2^\infty \frac{dw}{w(w + a)(w + 2)}$$
$$= \frac{16 \log(1 + a/2) - 8a \log 2}{4a - a^3} \qquad (16)$$

in agreement with a more general case, given in [3]. The values

$$C(0, 0) = 2 - \log 4 \qquad (17)$$

$$C(1, 0) = \frac{8}{3} \log \frac{9}{8} \qquad (18)$$

$$C(2, 0) = \log 2 - \frac{1}{2} \qquad (19)$$

$$C(4, 0) = \frac{1}{3} \log \frac{4}{3} \qquad (20)$$

$$C(6, 6) = \frac{1}{12} \log 2 \qquad (21)$$

entail only log 2 and log 3. This observation prompted the next step.

4 Dilogarithms at $b = 2$

By giving numerical evaluations to the lattice algorithm PSLQ, it was discovered that $C(a, 2)$ evaluates to dilogs with simple rational arguments, for $a \in \{1, 2, 4, 6\}$, namely

$$C(1, 2) = \pi^2 + 4\operatorname{Li}_2\left(\frac{1}{16}\right) - 8\operatorname{Li}_2\left(\frac{1}{6}\right)$$
$$- 16\operatorname{Li}_2\left(\frac{1}{4}\right) - 2\log^2 3 - 4\log^2 2 \qquad (22)$$

$$C(2, 2) = \frac{\pi^2}{12} - \operatorname{Li}_2\left(\frac{1}{4}\right) - \log^2 2 \qquad (23)$$

$$C(4, 2) = \frac{3}{8}\operatorname{Li}_2\left(\frac{1}{4}\right) + \frac{1}{8}\log^2 3 - \frac{3}{4}\log 2 \log \frac{3}{2} \quad (24)$$

$$C(6, 2) = \frac{2}{9}\operatorname{Li}_2\left(\frac{1}{4}\right) - \frac{1}{9}\operatorname{Li}\left(\frac{1}{16}\right) - \frac{1}{18}\log^2 2 \quad (25)$$

which indicated a dilogarithmic dependence of $C(a, 2)$ on a. Combining the superconvergence relation with the simplicity of $D(w, 2) = w^2$, a lengthy expression was proven by computer algebra, and then simplified by hand to give

$$\frac{1}{4} a^2 C(a, 2) = 3\mathrm{Li}_2(a/(a + 2)) - 2\mathrm{Li}_2(a/(2a + 4))$$
$$+ \mathrm{Li}_2(2a/(a - 2)) - \mathrm{Li}_2(a/(a - 2))$$
$$+ 2\mathrm{Li}_2(-a/4) + \log^2(1 + a/2)$$
$$- \log(1 - a^2/4) \log 2 \qquad (26)$$

which shows that $C(0, 2) = \log 2 - \frac{1}{2}$, in agreement with (19). Thanks to advice from Arttu Rajantie, it, became clear that the 5 dilogs could be simplified to give 2, using transformations of $\mathrm{Li}_2(x) := -\int_0^x (dy/y) \log(1 - y)$. The most compact formula is

$$\frac{1}{4} a^2 C(a, 2) = \mathrm{Li}_2((a - 2)/(a + 2)) - 2\mathrm{Li}_2(-2/(a + 2))$$
$$- \frac{1}{12} \pi^2 \qquad (27)$$

5 PSLQ and the symmetric tetrahedron

The previous results suggested the hypothesis that the totally symmetric tetrahedron, $C^{\mathrm{Tet}} := C(1, 1)$, is a dilogarithm. With the help of PSLQ, it was eventually reduced to a Clausen integral of startling simplicity:

$$\frac{C(1, 1)}{2^{5/2}} = -\int_{2\alpha}^{4\alpha} d\theta \log\left(2 \sin \frac{1}{2}\theta\right) \qquad (28)$$

with $\alpha := \arcsin \frac{1}{3}$. A proof appears to be rather difficult, though (28) has been confirmed numerically, at 1,000-digit precision. The discovery route was typical of work with PSLQ. Splitting $C(1, 1)$ into contributions below and above the 3-particle threshold, one finds that the latter involve terms of the form $\sqrt{2}\mathrm{Cl}_2(j\alpha + k\pi/6)$, with

$$\mathrm{Cl}_2(\theta) := \Im\mathrm{Li}_2(\exp(i\theta)) = \sum_{n>0} \frac{\sin(n\theta)}{n^2} \qquad (29)$$

and integer values of j and k. There appeared to be little prospect of reducing all terms to this set of constants, by analytical methods alone. Yet PSLQ found that the total is so reducible and also found many relations between such Clausen values and the constants $\{\pi \log 2, \pi \log 3, \alpha \log 2, \alpha \log 3\}$. As so often remarked in field theory, the whole:

$$\frac{C(1, 1)}{2^{5/2}} = \mathrm{Cl}_2(4\alpha) - \mathrm{Cl}_2(2\alpha) \qquad (30)$$

turned out to be far simpler than its parts. As a final bonus, this was transformed, again with the aid of PSLQ, to the exponentially convergent sum

$$C(1, 1) = \sum_{n=0}^{\infty} \frac{(-1/2)^{3n}}{n + \frac{1}{2}} \left(\frac{1}{n + \frac{1}{2}} - 3\log 2 - \sum_{m=1}^{n} \frac{3}{m}\right) \qquad (31)$$

formed from terms found in integer relations with $\sqrt{2}\mathrm{Cl}_2$ $(j\alpha + k\pi/6)$. This last result enables rapid computation in a single do-loop. The first 50 digits of

$$C^{\mathrm{Tet}} := C(1, 1) =$$
$$0.17390061066200274272650601711566596761380833829869 \qquad (32)$$

result in a trice, with 50,000 digits taking only 40 minutes on a 233 MHz Pentium. The first 1,000 digits agree with numerical quadrature of dispersive integrals, generously undertaken by Greg Fee, at CECM.

After this work was completed, Arttu Rajantie drew attention to an alternative representation of massive 3-dimensional tetrahedra [3], obtained by the method of differential equations [18]. In the totally symmetric case this gives [3]

$$\frac{C(1, 1)}{2^{5/2}} = \int_0^1 \frac{dx}{\sqrt{3 - x^2}} \left(\log \frac{3}{4} + \log \frac{3 + x}{2 + x}\right.$$
$$\left. - \frac{x^2}{4 - x^2} \log \frac{4}{2 + x} + \frac{x}{2 + x} \log \frac{3 + x}{3}\right) \qquad (33)$$

which appears to be no easier to reduce to (30) than the dispersive integral (2).

6 Conclusions

Thus PSLQ has shown that I was off target when suggesting at the recent Rheinsberg workshop that a super-renormalizable theory [1,3] might be less interesting, mathematically, than QCD [5]. In fact, the Ising tetrahedron is as intriguing as those in QCD.

One now sees that the symmetric 3-dimensional tetrahedron is given by (31) as an exponentially convergent sum that sits close to the classical formula [19]

$$\frac{\pi}{\sqrt{2}} = \sum_{n \geq 0} \frac{(-1/2)^n + (-1/2)^{3n+2}}{n + \frac{1}{2}} \qquad (34)$$

This association resonates strongly with the recent reduction [5] of a 4-dimensional tetrahedron, in the 3-loop QCD corrections to the electro-weak rho-parameter [20,21], to a sum of squares of two distinguished dilogarithms, namely $\zeta(2)$ and $\mathrm{Cl}_2(\pi/3)$. The latter was first encountered in 1-loop massless 3-point functions [22] and then in the pioneering work of van der Bij and Veltman [23] on 2-loop massive diagrams. In the massive case it appears in association with

$$\frac{\pi}{\sqrt{3}} = \sum_{n \geq 0} \frac{(-1/3)^n}{n + \frac{1}{2}} \qquad (35)$$

It remains to be seen whether the 'magic' connection proven in [24], between massless and massive instances of $\mathrm{Cl}_2(\pi/3)$, is generalizable to the quadrilogarithms found in [5] or to the dilogarithm (30) found here.

In conclusion: 3-loop single-scale vacuum diagrams in 4 dimensions [5] evaluate to quadrilogarithms of the sixth

root of unit, $\exp(i\pi/3) = (1 + i\sqrt{3})/2$, while in 3 dimensions we have now encountered dilogarithms of $\exp(i\alpha) = (\sqrt{8} + i)/3$. In both cases, there are remarkable transformations to exponentially convergent sums. In the 4-dimensional case, these entail polylogarithmic ladders, akin to those in [15], beginning with (35); in 3 dimensions (34) appears to provide the lowest rung. In both cases, the results are of a simplicity, scarcely to be expected from the method, that was revealed by PSLQ [6].

References

1. C. Gutsfeld, J. Küster, G. Münster, Nucl. Phys. B **479**, 654 (1996)
2. B.G. Nickel, D.I. Meiron, G.A. Baker, Univ. Guelph Report, 1977, diagram 12U4
3. A.K. Rajantie, Nucl. Phys. B **480**, 729 (1996); B **513**, 761 (1998)
4. D.J. Broadhurst, Z. Phys. C **47**, 115 (1990)
5. D.J. Broadhurst, hep-th/9803091, to appear in Eur. Phys. J. C
6. H.R.P. Ferguson, D.H. Bailey, S. Arno, http://science.nas.nasa.gov/Pubs/TechReports/NASreports/NAS-96-005
7. D. Kreimer, Phys. Lett. B **354**, 117 (1995); J. Knot Theory and its Ramifications **6**, 479 (1997); **7**, 61 (1998); *Knots and Feynman Diagrams*, Cambridge University Press, in preparation
8. D.J. Broadhurst, D. Kreimer, Int. J. Mod. Phys. C **6**, 519 (1995)
9. D.J. Broadhurst, R. Delbourgo, D. Kreimer, Phys. Lett. B **366**, 421 (1996)
10. D.J. Broadhurst, J.A. Gracey, D. Kreimer, Z. Phys. C **75**, 559 (1997)
11. D.J. Broadhurst, D. Kreimer, Phys. Lett. B **393**, 403 (1997)
12. D.J. Broadhurst, D. Kreimer, Phys. Lett. B **426**, 339 (1998)
13. D.J. Broadhurst, hep-th/9604128, to appear in J. Math. Phys.
14. J.-M. Borwein, D.M. Bradley, D.J. Broadhurst, Electronic J. Combinatorics **4**, R5 (1997)
15. D.J. Broadhurst, math.CA/9803067
16. J.M. Borwein, D.M. Bradley, D.J. Broadhurst, P. Lisonek, CECM reports 98-106, 98-107
17. T. Kinoshita, J. Math. Phys. **3**, 650 (1962); T.D. Lee, M. Nauenberg, Phys. Rev. B **133**, 1549 (1964)
18. A.V. Kotikov, Phys. Lett. B **254**, 158 (1991); Mod. Phys. Lett. A **6**, 677 (1991)
19. D.H. Bailey, P.B. Borwein, S. Plouffe, Mathematics of Computation **66**, 218 (1997)
20. L. Avdeev, J. Fleischer, S. Mikhailov, O. Tarasov, Phys. Lett. B **336**, 560 (1994)
21. K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, Phys. Lett. B **351**, 331 (1995)
22. W. Celmaster, R. Gonsalves, Phys. Rev. D **20**, 1420 (1979)
23. J.J. van der Bij, M. Veltman, Nucl. Phys. B **231**, 205 (1984)
24. A.I. Davydychev, J.B. Tausk, Phys. Rev. D **53**, 7381 (1996)